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THE SPHERICAL CENTERED COMPRESSION WAVE

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I. E. ZABABAKHIN and V. A. SIMONENKO
(Cheliabinsk)

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A solution is obtained for the problem of isentropic compression of perfect gas in a spherical centered wave, with infinite compression of the mass of initially homogeneous substance. Specific solutions are presented for adiabatic exponents equal 3 and $^5/_3$, and asymptotic formulas are derived for the general case. Obtained solutions are compared with known solutions.

Centered rarefaction waves with characteristics issuing from a single point represent a wide class of self-similar solutions of equations of gasdynamics in the one-dimensional plane case. A similar solution was derived by Staniukovich [1] for plane compression waves. Recently, considerable attention was given to the phenomenon of isentropic collapse, i.e. to the convergence of mass to a center. Asymptotics of pressure variation as a function of time was obtained for particles of matter subjected to such motions [2], and an analytic solution was given in [3, 4] for a new class of self-similar problems of isentropic collapse in the case of a particular profile of density in a compression wave.

A solution of the problem of the spherical centered compression wave contiguous to an initially quiescent matter is given below. In Fig. 1 curves 1, 2, and 3 represent, respectively, the particle "trajectory", the β-characteristics, and the τ-lines. The case involving collapse belongs to the class of self-similar solutions, referred to as quasi-simple spherical waves [5].

Let the matter undergoing compression be a perfect gas whose initial pressure, density, and speed of sound we denote, respectively, by p_0, ρ_0 and c_0. The adiabatic exponent is assumed to be $\gamma = 3$, since then the formulas are simpler (for other values of γ we shall present only certain final results). Note that similar results can be obtained for condensed matter whose equation of state is of the form $p = \gamma^{-1}\rho_0 c_0^2 [(\rho / \rho_0)^\gamma - 1]$ in which pressure is represented by the term $\gamma^{-1}\rho_0 c_0^2$.

In Riemann variables the equations of motion are of the form

$$\frac{\partial \alpha}{\partial t} + \alpha \frac{\partial \alpha}{\partial r} + \frac{\alpha^2 - \beta^2}{2r} = 0, \quad \frac{\partial \beta}{\partial t} + \beta \frac{\partial \beta}{\partial r} + \frac{\beta^2 - \alpha^2}{2r} = 0 \tag{1}$$

where $\alpha = u + c$ and $\beta = u - c$, and u and c are, respectively, the velocity of matter and the speed of sound. Their solution is sought in the self-similar form

$$\alpha(r, t) = \frac{r}{t} a(\tau), \quad \beta(r, t) = \frac{r}{t} b(\tau), \quad \tau = \frac{c_0 t}{r} \tag{2}$$

where t is the time measured from the instant of focusing and r is the distance from the center.

This investigation is similar to [5, 6], except for the choice of variables that are more convenient in this case. The substitution of parameters (2) into (1) yields

$$\tau \frac{da}{d\tau} = \frac{b^2 - 3a^2 + 2a}{2(1-a)}, \quad \tau \frac{db}{d\tau} = \frac{a^2 - 3b^2 + 2b}{2(1-b)} \tag{3}$$

The motion is bounded by a weak discontinuity $r = -c_0 t$ at which $u = 0$ and $c = c_0$. From this we derive for system (3) the initial conditions

$$a(-1) = -1, \quad b(-1) = 1$$

The problem consists of determining in the plane (a, b) the integral curve that joins the initial point A $(-1, 1)$ where $\tau = -1$ and the end point where $\tau = 0$.

The general pattern of curves in the plane (a, b) is shown in Fig. 2, where 1 denotes integral curves, 2 and 3 denote, respectively, isoclines of infinities and zeros, and 4 represents the loci at which the matter is brought to rest in a percussive manner $(a = b)$. Intersection of isoclines are singular points. Below, the following points are important: A $(-1, 1)$ and B $(1, 1)$ are directional nodes, O $(0, 0)$ is a diacritical node, S $((1 - \sqrt{3})/4, (1 + \sqrt{3})/4)$ is a saddle point with angular coefficients of separatrices $K_{1,2} = -4 \pm \sqrt{15}$.

At the initial point A parameter $\tau = -1$ which increases along the curve and at point S vanishes. Denoting $a - a_S = x$ and $b - b_S = y$ and assuming that close to S $y = kx$, after the rejection of small terms in Eqs. (3) and simplifications, we obtain

$$\tau\, dx / d\tau = (\sqrt{5} - 1)\, x$$

hence $\tau = x^{0.809}$ and when $x \to 0$ also $\tau \to 0$. Thus the separatrix AS of saddle S is the sought curve and point S corresponds to a collapse.

The whole curve AS is determined by numerical integration of system (3) (from S to the left), after which the numerical integration of equation $u = (\alpha + \beta)/2$ or $dr/dt = r(a + b)/(2t)$ yields the piston law of motion $r(t)$ and, then the speed $c = r(a - b)/(2t)$, and pressure $p/p_0 = (c/c_0)^3$ of sound.

Close to the focus

$$\frac{dr}{dt} \approx \frac{(a_S + b_S) r}{2t} = \frac{r}{4t}, \quad c = \frac{(a_S - b_S) r}{2t} = -\frac{\sqrt{3} r}{4t}$$

i.e. $r \sim t^{1/4}$ and $p \sim c^3 \sim (r/t)^3 \sim t^{-9/4}$.

For other γ (without adducing the derivation)

$$a_S = (1 - \sqrt{\nu})\,\eta, \quad b_S = (1 + \sqrt{\nu})\,\eta, \quad \eta = 2/(\nu\gamma - \nu + 2) \tag{4}$$

$$r \sim t^{\eta}, \quad p \sim t^{-\nu\gamma\eta}, \quad E^{\cdot} \sim t^{-\zeta}, \quad \zeta = [1 + {}^3/_2\, \nu(\gamma - 1)]\,\eta$$

where $\nu = 1, 2, 3$, respectively, for the plane, cylindrical and spherical cases, $E^{\cdot}(t)$ is the energy flux per unit of time through a selected spherical surface of the matter. These formulas are in agreement with related formulas in [2 – 4, 7] but differ from those in [8].

The dependence of pressure on the piston p/p_0 and of the mean compression of the sphere $\overline{\delta} = (r_0/r)^3$ on time t/t_0 (t_0 is the total compression time) for $\gamma = 3$ and $\gamma = {}^5/_3$ is shown in Fig. 3 by curves 1 and 2, respectively. Curves 3 and 4, which are shown there for comparison, relate to results of calculations by asymptotic formulas

(4) that are used (see, e.g., [2]) in applied calculations. The coefficients in asymptotic formulas were numerically computed.

Let us explain the meaning of other integral curves in the neighborhood of AS. The curves lying above the saddle point S are not solutions of the problem, since the derivative $db / d\tau$ changes its sign at intersection with the horizontal $b = 1$, i.e. τ begins again to decrease, without tending to zero, and the solution for a and b becomes two-valued. Other curves of the SMO (*) type correspond to real motions, i.e. to quasi-simple waves [5]. At point O of these curves τ vanishes, and close to that point from (3) we have $\tau da / d\tau = a$, i.e. $\tau = ma$ and $\tau \to 0$ when $a \to 0$. The velocity remains constant, since close to zero $b = na$ and

$$u = \frac{(a+b)r}{2t} \approx \frac{c_0(1+n)}{2m} \tag{5}$$

The pattern of motion for that case is shown in Fig. 4 (OB is the reflected wave; the remaining notation conforms to Fig. 1). The compression wave is not centered (the characteristics do not converge at the center) but the similarity (self-similarity) of particle trajectories remains. The velocity at instant $t = 0$ is everywhere the same and directed toward the center, which corresponds to the initial state in Sedov's solution of the problem of gas focusing at a point [6]. It is defined by the continuation of curves beyond the point O up to intersection with line MN. Shock wave OB issues from the center, behind which the gas is at rest.

The extreme lower curve AO in Fig. 2 corresponds to quiescence and absence of compression ($a = -b$, $u = 0$), while the upper one which adjoins SO corresponds to infinitely high velocity and compression. The latter is clear, since the variation of τ along a curve close to SO is slow (at S and O $\tau = 0$), hence $m = 0$ and by (5) velocity u is infinitely high. It is, consequently, possible to formulate the laws of motion of a particular particle and of pressure variation in it, laws that define any compression of an initially homogeneous sphere from 1 to ∞ with also homogeneous final states.

Solutions that correspond to integral curves within the curvilinear triangle BSO in Fig. 2 and their continuations to intersection with MN are of a similar character. Contraction of the envelope at whose inner boundary $c = 0$ and $u = \text{const}$ corresponds to these physically. Line BS corresponds to collapse. The asymptotic variation of parameters is then the same as in the case of a cylindrical wave [4].

Thus flows of the type of centered compression waves may be relaized in initially homogeneous spherical systems. A finite mass of matter then undergoes infinite compression and the energy of compression also tends to be infinitely great.

As was done in [7], it is possible to indicate other solutions that have the same properties, for instance, when the self-similar variable $\tau \sim t / r^q$ $(q > 1)$ is a power function. However such solutions correspond to specially created profiles that are physically artificial. Some of these may be considered as limits of real motions [7]. Results of this investigation supplement those in [5] for $q = 1$.

The asymptotic formulas obtained here for the variation of gasdynamic quantities are the same as the solutions presented in [3, 4]. But unlike the cases considered in those investigations, the solution presented here for the centered wave, as well as for spherical

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*) Editor's note. Self-similar Motions of Object.

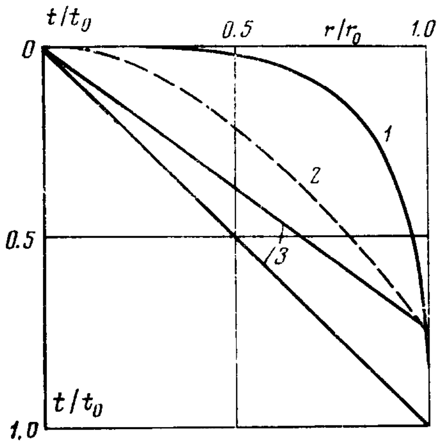

Fig. 1

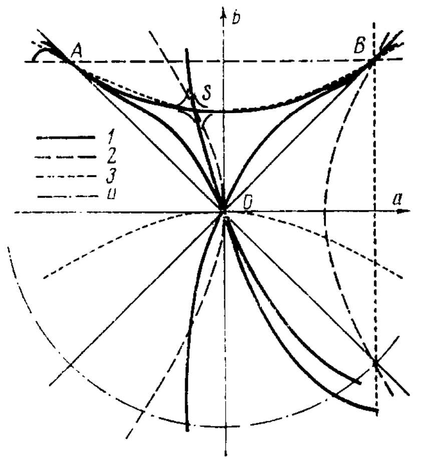

Fig. 2

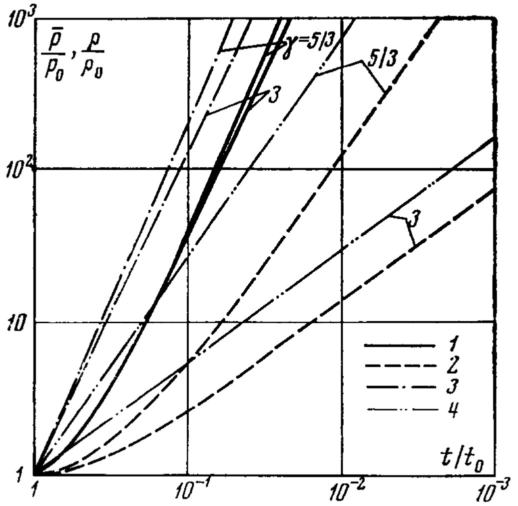

Fig. 3

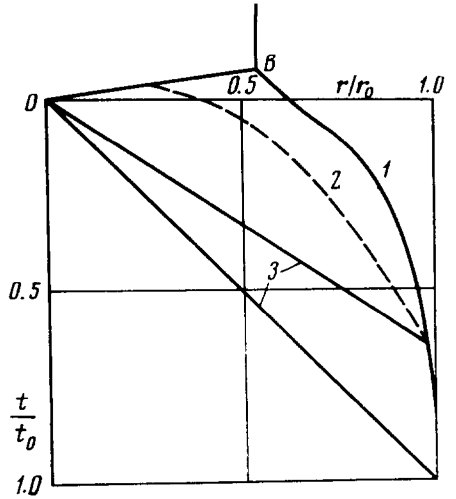

Fig. 4

quasi-simple waves of similar character, does not require the creation of special profiles of parameters. This means that one can hope to realize experimentally the results obtained here, similar to those in [2, 8].

REFERENCES

1. Staniukovich, K.P., Unsteady Motion of Continuous Media. (English translation), Pergamon Press, Book No. 09320, 1960.
2. Clarke, J., Fisher, H.M. and Mason, R.J., Laser-driven implosion of spherical DT targets to thermonuclear burn conditions. Phys. Rev., Letters. Vol. 30, No. 3, 1973.
3. Kidder, R.E., Theory of homogeneous isentropic compression and its applications to laser fusion. Nucl. Fusion, Vol. 14, No. 1, 1974.
4. Zmitrienko, N.V. and Kurdiumov, S.P., Self-similar compression mode of a finite plasma mass. Dokl. Akad. Nauk SSSR, Vol. 218, No. 6, 1974.
5. Courant, R. and Friedrichs, K., Supersonic Flow and Shock Waves. Moscow, Izd. Inostr. Lit., 1950.
6. Sedov, L.I., Methods of Similarity and Dimensionality in Mechanics. Moscow, Gostekhizdat, 1954.
7. Godunov, S.K. and Kireeva, I.L., On certain self-similar motions of perfect gas. (English translation), Pergamon Press. Zh. Vychisl. Mat. i mat. Fiz., Vol. 8, No. 2, 1968.
8. Nuckolls, J., Wood, L., Thiessen, A. and Zimmerman, G., Laser compression of matter to super-high densities. Thermonuclear (CTR) applications. Nature, Vol. 239, No. 5368, 1972.

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